Augmented Recursion For One-loop Amplitudes

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We present a semi-recursive method for calculating the rational parts of one-loop amplitudes when recursion produces double poles. We illustrate this with the graviton scattering amplitude $M^{\text{1-loop}}(1^-,2^+,3^+,4^+,5^+)$.

1. Introduction

On-shell recursive techniques, using the rationality of tree amplitudes and their complex factorisation properties, have proven very successful in the computation of scattering amplitudes in gauge and gravity theories [1, 2]. Specifically, in a theory with massless states, if we use a spinor helicity representation for the polarisation vectors it is possible to write the amplitude entirely in terms of spinorial variables $A(\lambda^i_\alpha, \bar\lambda^i_{\dot\alpha})$ where the massless momentum of the i^{th} particle is $\lambda^i_\alpha\bar\lambda^i_{\dot\alpha} = (\sigma_\mu)_{\alpha\dot\alpha}k^\mu_i$. The analytic structure of the amplitude can be probed by choosing a pair a, b of external momenta and shifting these according to

$$\bar\lambda^a \longrightarrow \bar\lambda^a - z\bar\lambda^b, \quad \lambda^b \longrightarrow \lambda^b + z\lambda^a \tag{1}$$

where we suppress the spinor indices. If the shifted amplitude $A(z)$ (a) is a rational function, (b) has finite order poles at points z_i, and (c) vanishes as $z \longrightarrow \infty$, then applying Cauchy's theorem to $A(z)/z$ with a contour at infinity yields

$$A(0) = -\sum_{\text{poles } z_i} \operatorname*{Res}_{z=z_i} \frac{A(z)}{z}. \tag{2}$$

At tree level the factorisation of amplitudes is simple: amplitudes must factorise on multi-particle and collinear poles into the product of two tree amplitudes defined at $z = z_i$. Thus we can express the n-point tree amplitude in terms of lower point amplitudes,

$$A^{\text{tree}}_n(0) \;=\; \sum_{i,\sigma} A^{\text{tree},\sigma}_{r_i+1}(z_i)\frac{i}{K^2}A^{\text{tree},-\sigma}_{n-r_i+1}(z_i), \tag{3}$$

where the summation over i is only over factorisations where the a and b legs are on opposite sides of the pole. This technique is very effective in computing tree amplitudes and has been extended to a variety of other applications including gravity [2].

Beyond tree level there are three potential barriers to using recursion. Firstly, the amplitudes generally contain non-rational functions such as logarithms and dilogarithms; secondly, the amplitudes may contain higher-order poles for complex momenta; and finally, the amplitudes may not vanish asymptotically with z. Nonetheless a variety of techniques based upon recursion and unitarity have been developed. A one-loop amplitude for massless particles may be expressed as

$$A^{\text{1-loop}} = \sum_{n=2,3,4;i} c_i I^i_n + R, \tag{4}$$

where the scalar integral functions I^i_n are the various scalar box, triangle and bubble functions. The amplitude can thus be determined by computing the rational coefficients, c_i, and the purely rational term R. The c_i can be computed by the four-dimensional unitarity technique [3–5] or indeed recursively [6]. Many techniques have been developed for evaluating R: D-dimensional unitarity, recursion and specialised Feynman diagram techniques [7–20].

In general, the rational term R does not simply satisfy the requisites for recursion. If the amplitude has only simple poles but does not vanish as $z \longrightarrow \infty$ then it can be possible to for-

mulate auxiliary recursion relations [21]. However there are rational amplitudes for which one cannot find a shift which only generates simple poles such as the single-minus amplitudes $A^{\text{1-loop}}(1^-,2^+,\cdots,n^+)$. These amplitudes vanish at tree level and consequently are purely rational at one-loop. A shift on these amplitudes yields double and single poles. The double pole is *not* in itself a a barrier to using recursion, however to obtain the full residue one needs to know the coincident single pole, or the 'pole under the double pole', which is not determined by factorisation into on-shell amplitudes. In [22] for Yang–Mills this was postulated to be

$$\frac{1}{(K^2)^2}+\frac{S(a_1,\hat{K}^+,a_2)S(b_1,\hat{K}^-,b_2)}{K^2}, \tag{5}$$

where the 'soft' factors are $S(a,s^+,b) = \langle a\,b\rangle/(\langle a\,s\rangle\,\langle s\,b\rangle)$, $S(a,s^-,b) = [a\,b]/([a\,s]\,[a\,b])$, and a_1, a_2 (b_1, b_2) are colour-adjacent to K on the left (right) side of the pole. With this ansatz, recursion correctly reproduces the known single-minus one-loop amplitudes. In [23] it was shown that the consistency requirements for recursion in QCD are sufficient to determine these soft factors.

The above postulate, or variations thereof, does not work for gravity amplitudes [24]. Here, we apply a semi-recursive technique for gravity scattering amplitudes that obtains the 'pole under the pole' using an axial gauge formalism to calculate the previously-unknown amplitude $M^{\text{1-loop}}(1^-,2^+,3^+,4^+,5^+)$. We assume that the shifted amplitudes vanish as $z\longrightarrow\infty$. The derived amplitude has the correct symmetries and soft limits, providing strong evidence for the validity of this assumption. Further, we have checked the result by a completely independent 'string-based rules' [25, 26] computation.

2. Recursion

The factorisation of one-loop massless amplitudes is described in [27],

$$A_n^{\text{1-loop}}\longrightarrow\sum\Big[A_{r+1}^{\text{1-loop}}\,\frac{i}{K^2}\,A_{n-r+1}^{\text{tree}} + A_{r+1}^{\text{tree}}\frac{i}{K^2}A_{n-r+1}^{\text{1-loop}}+A_{r+1}^{\text{tree}}\frac{i}{K^2}A_{n-r+1}^{\text{tree}}F_n\Big], \tag{6}$$

where the one-loop 'factorisation function' F_n is helicity-independent. Naïvely this only contains single poles, however for complex momenta there are double poles. These can be interpreted as due to the three-point all-plus (or all-minus) one-loop amplitude also containing a pole

$$A_3^{\text{1-loop}}(K^+,a^+,b^+)=\frac{1}{K^2}V_3^{\text{1-loop}}(K^+,a^+,b^+) \tag{7}$$

where, for pure Yang–Mills,

$$V_3^{\text{1-loop}}(K^+,a^+,b^+)=-\frac{i}{48\pi^2}\,[K\,a]\,[a\,b]\,[b\,K]\,. \tag{8}$$

Explicitly, consider the amplitude [28]:

$$A_5^{\text{1-loop}}(1^-,2^+,3^+,4^+,5^+)\sim\frac{1}{\langle 3\,4\rangle^2}\Bigg[-\frac{[2\,5]^3}{[1\,2]\,[5\,1]} + \frac{\langle 1\,4\rangle^3\,[4\,5]\,\langle 3\,5\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle^2}-\frac{\langle 1\,3\rangle^3\,[3\,2]\,\langle 4\,2\rangle}{\langle 1\,5\rangle\,\langle 5\,4\rangle\,\langle 3\,2\rangle^2}\Bigg]. \tag{9}$$

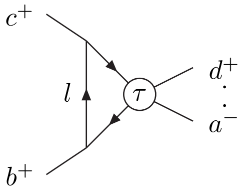
If we carry out a complex shift on λ_5,$\bar\lambda_1$ as in eq. (1) then $\langle 4\,5\rangle\longrightarrow\langle 4\,5\rangle+z\,\langle 4\,1\rangle$ which vanishes at $z=-\langle 4\,5\rangle/\langle 4\,1\rangle$ and the amplitude has a double pole at this point.

Computing this amplitude using $V_3^{\text{1-loop}}$ correctly generates the double pole in the amplitude [22, 24], however it needs augmentation to give an expression with the correct single pole. By trial and error, adding the second term in (5) gives the correct single pole and completes the computation of the amplitude.

For gravity the vertex

$$V^{\text{1-loop}}(K^+,a^+,b^+)=-\frac{i\kappa^3([K\,a]\,[a\,b]\,[b\,K])^2}{1440\pi^2} \tag{10}$$

can be used to generate a double pole term but attempts [24] to implement a universal correction for the single pole analogous to that of (5) have failed. The resolution is to replace the factorisation term of (6) with a tree insertion diagram:



which we compute using axial gauge diagrammatics. The circle in the diagram represents the sums of all possible tree diagrams with two internal legs and the given external legs, which we denote τ. Note that we evaluate these diagrams for real momenta and only carry out analytic shifts on the final expressions.

3. Axial gauge diagrammatics

Following [29] we use a set of Feynman rules for Yang–Mills amplitudes based on scalar propagators connecting three and four point vertices. The starting point is the expansion of the axial gauge propagator in terms of polarisation vectors,

$$i\frac{d_{\mu\nu}}{k^2} = \frac{i}{k^2}[\epsilon^+_\mu(k)\epsilon^-_\nu(k)+\epsilon^-_\mu(k)\epsilon^+_\nu(k)+\epsilon^0_\mu(k)\epsilon^0_\nu(k)], \tag{11}$$

where

$$\epsilon^+_\mu = \frac{[k^\flat|\gamma_\mu|q\rangle}{\sqrt{2}\,\langle k^\flat\, q\rangle}, \epsilon^-_\mu = \frac{[q|\gamma_\mu|k^\flat\rangle}{\sqrt{2}\,[k^\flat\, q]}, \epsilon^0_\mu = 2\frac{\sqrt{k^2}}{2k\cdot q}q_\mu, \tag{12}$$

with

$$k^\flat := k - \frac{k^2}{2k\cdot q}q, \tag{13}$$

where q is a null reference momentum which may be complex. The resulting three-point vertices are,

$$\begin{aligned} \frac{1}{i\sqrt{2}}V_3^{\mathrm{MHV}}(1^-,2^-,3^+) &= \frac{\langle 1\, 2\rangle\,[3\, q]^2}{[1\, q]\,[2\, q]}, \\ \frac{1}{i\sqrt{2}}V_3^{\overline{\mathrm{MHV}}}(1^+,2^+,3^-) &= \frac{[2\, 1]\,\langle 3\, q\rangle^2}{\langle 1\, q\rangle\,\langle 2\, q\rangle}, \end{aligned} \tag{14}$$

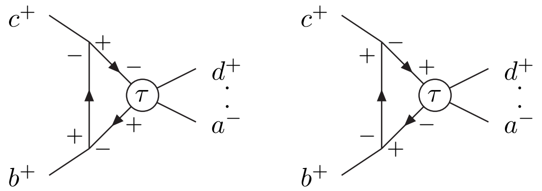
along with a $V_3(1^+,2^-,3^0)$ vertex which can be absorbed into effective four-point vertices.

When adopting a recursive approach which involves shifting a negative-helicity leg a and a positive-helicity leg b, the recursion-optimised choice for the reference momentum q is

$$\lambda_q = \lambda_a, \qquad \bar\lambda_q = \bar\lambda_b. \tag{15}$$

With this choice of q the leg a (b) can only enter a diagram on a V_3^{MHV} ($V_3^{\overline{\mathrm{MHV}}}$) vertex, and there are no four-point vertices in the single-minus amplitudes at tree or one-loop level.

Singularities arise in the loop integration from the region of loop momentum where the denominators of three adjacent propagators vanish simultaneously. This requires the two null legs to which the propagators connect to become collinear. In the integration region of interest all the legs of τ are close to null and τ approaches the corresponding on-shell tree amplitude. The internal legs are also close to collinear. Helicity configurations for which τ is singular in this collinear limit, shown in Fig. 1, contribute to the double (and single) pole, conversely those that give a vanishing τ in the collinear limit give no residue.



(a) (b)

Figure 1: Contributing helicity structures

The diagram of Fig. 1(b) evaluates to

$$\int d^4l\, \frac{[b|l|a\rangle[c|l|a\rangle}{\langle b\, a\rangle\,\langle c\, a\rangle}\frac{\langle C\, a\rangle^2}{\langle B\, a\rangle^2}\frac{\tau(C^+,d^+,\cdots,a^-,B^-)}{l^2(l+k_b)^2(l-k_c)^2} \tag{16}$$

where $B = l+b$, $C = c-l$, and the momenta in the spinor products are q-nullified as in (13). We construct a basis for the loop momentum using b and c:

$$\begin{aligned} &l = \alpha_1(k_b+k_c) + \alpha_2(k_b-k_c) + \\ &(\alpha_3+i\alpha_4)\frac{\langle c\, a\rangle}{\langle b\, a\rangle}\lambda_b\bar\lambda_c + (\alpha_3-i\alpha_4)\frac{\langle b\, a\rangle}{\langle c\, a\rangle}\lambda_c\bar\lambda_b \end{aligned} \tag{17}$$

Under this parametrisation,

$$\int \frac{d^4l\ \ f(l)}{l^2(l+k_b)^2(l-k_c)^2} = \frac{1}{s_{bc}} \int d\alpha_i\ F(\alpha_i) f(l(\alpha_i)) \tag{18}$$

where $F(\alpha_i)$ has no dependence on s_{bc}. The integrand from Fig. 1(b) then becomes,

$$\frac{[bc]}{\langle bc\rangle} \frac{\langle C\,a\rangle^2}{\langle B\,a\rangle^2} \tau(C^+, d^+, \ldots, a^-, B^-) \times F'(\alpha_i). \tag{19}$$

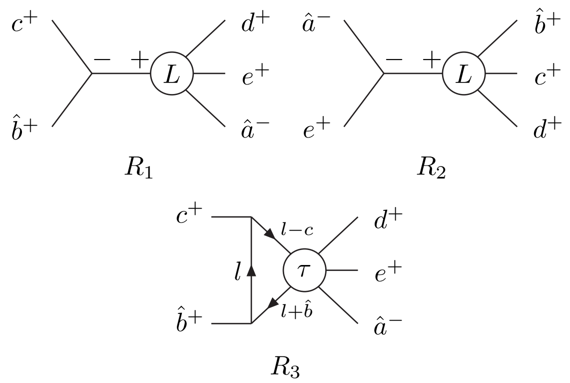
In order to evaluate the contribution from (19) we must evaluate the tree structures to order $\langle b\,c\rangle^0$. For diagrams within τ involving $1/s_{bc}$, this means going beyond leading order. These correspond to triangles in the full diagram and the calculation is readily done exactly. The diagrams without this propagator need only be calculated to leading order. In this regard, not only is the recursive approach selecting a subset of diagrams for calculation, it is also allowing us to calculate these diagrams in a very convenient limit.

For gravity the equivalent expression to (19) is

$$\frac{[bc]^3}{\langle bc\rangle} \frac{\langle C\,a\rangle^4}{\langle B\,a\rangle^4} \tau_{\rm g}(C^+, d^+, \ldots, a^-, B^-) \times \tilde{F}(\alpha_i). \tag{20}$$

4. The graviton scattering amplitude $M^{\textbf{1-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$

There are three types of recursive contribution to this amplitude, which in turn are summed over the distinct permutations of c, d and e. Diagrams R_1 and R_2 involve only single poles and are obtained from the corresponding four-point one-loop amplitudes for the circles marked L.



Doing recursion with the shift (1), we obtain

$$R_1(a,b,c,d,e) = \frac{1}{5760} \frac{\langle ad\rangle^2 \langle ae\rangle^2 [bc][de]^4}{\langle ab\rangle^2 \langle bc\rangle \langle ce\rangle^2 \langle cd\rangle^2 \langle de\rangle^2} \times \left(\langle cd\rangle^2 \langle ae\rangle^2 + \langle ac\rangle \langle cd\rangle \langle de\rangle \langle ae\rangle + \langle ac\rangle^2 \langle de\rangle^2\right), \tag{21}$$

$$R_2(a,b,c,d,e) = -\frac{3}{5760} \frac{\langle ae\rangle [be]^4}{\langle cd\rangle^2 [ab]^2 [ae]} \times \left([bc]^2 [de]^2 + [bc][cd][de][be] + [cd]^2 [be]^2\right). \tag{22}$$

Diagram R_3 contains a double pole so we must evaluate $\tau_{\rm g}$ of (20). We use the five-point KLT relation [30],

$$M(a^- B^- C^+ d^+ e^+) = s_{BC} s_{de} A(a^- B^- C^+ d^+ e^+) A(a^- C^+ B^- e^+ d^+) + s_{Bd} s_{Ce} A(a^- B^- d^+ C^+ e^+) A(a^- d^+ B^- e^+ C^+) \tag{23}$$

in a form that restricts the $\langle b\,c\rangle$ pole to the first term. We calculate this as Laurent series in $\langle b\,c\rangle$, dropping terms that will not contribute to the residues. While the KLT relations are only valid for on-shell momenta, we assume the deviation of (23) from a direct off-shell calculation may be neglected [1] in the region around $B^2 = C^2 = 0$.

In our choice of axial gauge, $A(a^- B^- C^+ d^+ e^+)$ receives contributions from five diagrams, only two of which contain a $V_3(B^-, C^+, x)$ vertex and thus contribute to τ's collinear singularity. De-

[1] The general case is worthy of further study [31].

noting these by D_a and D_b, we find, using (17)

$$D_a + D_b = \frac{\langle Ba\rangle^2}{\langle Ca\rangle^2}\frac{\langle a|bc|a\rangle}{s_{bc}[ab]\langle da\rangle\langle ea\rangle}\times \left(\frac{[b|ad|e]-[b|cb|e]}{[ae]\langle de\rangle}\right) f_a(\alpha_i), \quad (24)$$

where $f_a(\alpha_i)$ is some function that depends only on the integral parameters, α_i. We note that the second term is sub-leading in the $\langle b\,c\rangle$ pole.

The leading pole in $A(a^- C^+ B^- e^+ d^+)$ is obtained similarly and we obtain the full leading $\langle b\,c\rangle$ pole in (23) as

$$\frac{\langle Ba\rangle^4}{\langle Ca\rangle^4} s_{bc} s_{de} \frac{\langle ab\rangle^2\langle ac\rangle^2[de]^3[bc]}{\langle bc\rangle\langle de\rangle[a|d+e|a\rangle} f'_a(\alpha_i). \quad (25)$$

Combining this with the factors arising from the left hand part of the full diagram (*cf.* (20)) and integrating over the α_i the leading term in the Laurent series for R_3 is proportional to

$$\frac{[bc]^4\langle ab\rangle^2\langle ac\rangle^2[de]^3}{\langle bc\rangle^2\langle de\rangle[a|d+e|a\rangle} \equiv \mathcal{D}, \quad (26)$$

which clearly displays the double pole factor.

We now express each sub-leading contribution to (20) as $\mathcal{D}\times\delta_j f_j(\alpha_i)$. Firstly there is the sub-leading contribution of (24) together with the corresponding contribution from $A(a^- C^+ B^- e^+ d^+)$:

$$\delta_1 = \frac{s_{bc}[be]}{[b|ad|e]} + \frac{s_{bc}[bd]}{[b|ae|d]}. \quad (27)$$

The remaining diagrams for $A(a^- B^- C^+ d^+ e^+)$ (and its counterpart $A(a^- C^+ B^- e^+ d^+)$), in which B and C enter on different vertices contribute

$$\delta_2 = \frac{s_{bc}[e|a|c\rangle}{s_{ab}[e|d|c\rangle}, \quad (28)$$

$$\delta_3 = \frac{\langle bc\rangle\langle de\rangle}{s_{ab}[de]}\left(\frac{[e|B|a\rangle[eb]}{\langle da\rangle\langle cd\rangle} + \frac{[d|B|a\rangle[db]}{\langle ea\rangle\langle ce\rangle}\right). \quad (29)$$

These diagrams are finite in the collinear limit, so we can drop terms proportional to B^2 and C^2. Finally we need the second term in (23), which is also finite in the collinear limit and can be evaluated using MHV tree amplitudes, yielding:

$$\delta_4 = \frac{\langle bc\rangle\langle de\rangle[d|B|a\rangle[e|C|a\rangle}{[bc][de]\langle ab\rangle^2\langle cd\rangle\langle ce\rangle}. \quad (30)$$

Up to $\mathcal{O}(\langle b\,c\rangle^{-1})$ (20) is then expressed as

$$\frac{[bc]^4\langle ab\rangle^2\langle ac\rangle^2[de]^3}{\langle bc\rangle^2\langle de\rangle[a|d+e|a\rangle}\left(1+\sum_j \delta_j f_j(\alpha_i)\right) F'(\alpha_i). \quad (31)$$

This has purely polynomial dependence on the α_i. The integration thus gives constant numerical factors which may be obtained by direct evaluation or, more conveniently, by considering collinear limits.

We now determine the amplitude recursively by applying the shift (1) to the integrated (31) and evaluating the residue at $z = -\langle bc\rangle/\langle ac\rangle$. The coefficient of the double pole has a z dependence under this shift which generates a further contribution to the single pole since

$$\operatorname*{Res}_{z=z_i} \frac{f(z)}{z(z-z_i)^2} = -\frac{f(z_i)}{z_i^2} + \frac{1}{z_i}\left.\frac{df}{dz}\right|_{z=z_i}. \quad (32)$$

The full contribution from R_3 is then

$$R_3(a,b,c,d,e) = \frac{1}{5760}\frac{\langle ab\rangle^2\langle ac\rangle^4[bc]^4[de]}{\langle ad\rangle\langle ae\rangle\langle bc\rangle^2\langle cd\rangle\langle ce\rangle\langle de\rangle} \times\big(1+\Delta(a,b,c,d,e)\big), \quad (33)$$

where

$$\Delta(a,b,c,d,e) = -\frac{1}{2}\frac{\langle ad\rangle\langle bc\rangle}{\langle ab\rangle\langle cd\rangle} - \frac{1}{2}\frac{\langle ae\rangle\langle bc\rangle}{\langle ab\rangle\langle ce\rangle} - 3\frac{[db][eb]\langle bc\rangle\langle de\rangle}{\langle dc\rangle\langle ec\rangle[bc][de]} - 3\frac{[dc][ec]\langle bc\rangle\langle de\rangle\langle ca\rangle^2}{\langle dc\rangle\langle ec\rangle[bc][de]\langle ba\rangle^2} - \frac{7}{2}\frac{[dc][eb]\langle bc\rangle\langle de\rangle\langle ca\rangle}{\langle dc\rangle\langle ec\rangle[bc][de]\langle ba\rangle} - \frac{7}{2}\frac{[db][ec]\langle bc\rangle\langle de\rangle\langle ca\rangle}{\langle dc\rangle\langle ec\rangle[bc][de]\langle ba\rangle}. \quad (34)$$

The full amplitude is the sum over contributions arising from three orderings of external legs,

$$M^{\text{1-loop}}(1^-,2^+,3^+,4^+,5^+) = R(1,2,3,4,5) + R(1,2,4,5,3) + R(1,2,5,3,4), \quad (35)$$

(the full amplitude has a factor of $i\kappa^5/16\pi^2$), and each R is the sum of the recursive diagrams,

$$R = R_1 + R_2 + R_3. \quad (36)$$

This expresion has the correct collinear limits, is symmetric under interchange of pairs of positive-helicity legs and agrees numerically with that calculated by string-based rules. We have also calculated $M^{\text{1-loop}}(1^-, 2^+, 3^+, 4^+, 5^+, 6^+)$ [32], and again checked that it has the correct symmetries and collinear limits. *Mathematica* code for the five- and six-point amplitudes may be found at `http://pyweb.swan.ac.uk/~dunbar/graviton.html`.

5. Conclusions and remarks

We have demonstrated how to augment recursion to determine the rational terms in amplitudes with double poles under a complex shift. Double poles are unavoidable in the case of the amplitudes $A^{\text{1-loop}}(1^-, 2^+, 3^+, \ldots, n^+)$ in both Yang-Mills and gravity. In the absence of a universal soft factor analogous to (5), to perform the augmented recursion the sub-leading poles must be determined on a case-by-case basis. While we have done this for both the five- and six-point single-minus gravity amplitudes, this procedure could be used to calculate any higher-point single-minus amplitude.

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